A FORMULA FOR IDEAL LATTICES OF GENERAL COMMUTATIVE RINGS

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Abstract

Let S be a set of n ideals of a commutative ring A and let G_{even} (respectively G_{odd}) denote the product of all the sums of even (respectively odd) number of ideals of S. If $n \leq 6$ the product of G_{even} and the intersection of all ideals of S is included in G_{odd}. In the case A is an Noetherian integral domain, this inclusion is replaced by equality if and only if A is a Dedekind domain.

Key words: GCD, LCM, ideal lattice, Dedekind domain

1. Introduction

We know that the product of the greatest common divisor (GCD) and the least common denominator (LCM) of two natural numbers a and b is the product ab. Further, it is known that, given a finite set of natural numbers, their GCD is expressed in terms of LCMs of its subsets and that their LCM is expressed in terms of GCDs of its subsets. (see Wolfram [4], [5]). These can be generalized to GCDs and LCMs of ideals of a Dedekind domain as follows.

The set of ideals of a commutative ring form a lattice with respect to the order of inclusion. The GCD and the LCM of a finite number of ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ are defined to be the upper bound

$$\mathfrak{a}_1 \vee \cdots \vee \mathfrak{a}_n = \mathfrak{a}_1 + \cdots + \mathfrak{a}_n$$

and the lower bound

$$\mathfrak{a}_1 \wedge \cdots \wedge \mathfrak{a}_n = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n$$

of them. Let S be a set of ideals of a Dedekind domain with n elements. If $1 \leq k \leq n$, $G(k)$ (respectively $L(k)$) denotes the product of all GCDs (respectively LCMs) of subsets of S with k elements. Then we have the following equalities .

$$(*)_n \qquad G(n)L(2)L(4)\cdots L(2\lfloor n/2 \rfloor) = L(1)L(3)\cdots L(2\lceil n/2 \rceil - 1),$$

$$(**)_n \qquad L(n)G(2)G(4)\cdots G(2\lfloor n/2 \rfloor) = G(1)G(3)\cdots G(2\lceil n/2 \rceil - 1).$$

Conversely, if A is a Noetherian integral domain, even one simple equality

$$(*)_2 = (**)_2 \qquad (\mathfrak{a} \wedge \mathfrak{b})(\mathfrak{a} \vee \mathfrak{b}) = \mathfrak{a} \cdot \mathfrak{b}$$

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for any pair $\mathfrak{a}$ and $\mathfrak{b}$ of ideals of a Noetherian integral domain A implies that it is a Dedekind domain. These are proved in §3.

Both of these equalities fail in the case of general commutative rings as seen in §4. We found however, if we replace equalities by inclusions, one of them remains valid for $n \le 6$ ideals. This is our main result:

$$(\dagger)_n \qquad L(n)G(2)G(4)\cdots G(2\lfloor n/2\rfloor) \subset G(1)G(3)\cdots G(2\lceil n/2\rceil - 1).$$

The number n of the ideals is restricted at present. The case of at most five ideals is proved by ordinary mathematical reasoning (see **5.1**) and the case of six ideals using a computer program in §6.

2. Some combinatorial formulae

Let $\tilde{T}$ denote the set of all finite list (sequence) of elements of a totally ordered set T permitting repetition and $\bar{T}$ the quotient of $\tilde{T}$ identifying permutated lists. In other words, an element of $\bar{T}$ is a *multiset* of elements of T. In case of an extensional expression of a multiset, we use square parenthesis "[" and "]". Hence, if $\alpha_i \ne \alpha_j$ $(i \ne j)$, the multiset $[\alpha_1, \ldots, \alpha_n]$ can be identified with the set $\{\alpha_1, \ldots, \alpha_n\}$. Let us put

$$N(n) := \{1, 2, \ldots, n\}.$$

For a multiset $[\alpha_1, \ldots, \alpha_n]$ of elements of T, we define the multisets

$$\overline{S}_k := \overline{S}_k(\alpha_1, \ldots, \alpha_n) := \Big[\alpha_{i_1} \vee \cdots \vee \alpha_{i_k} :\ (i_1, \ldots, i_k) \in \binom{N(n)}{k}\Big],$$

$$\underline{S}_k := \underline{S}_k(\alpha_1, \ldots, \alpha_n) := \Big[\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} :\ (i_1, \ldots, i_k) \in \binom{N(n)}{k}\Big],$$

where $\binom{N(n)}{k}$ stands for the family of all subsets with k distinct elements of $N(n)$ and $\vee$ and $\wedge$ express the supremum and the infimum with respect to the total order. We can define the join $\cup$ of elements of $\bar{T}$ in an obvious manner.

Lemma 2.1. *Let T be a totally ordered set. For a multiset $[\alpha_1, \ldots, \alpha_n]$ of elements of T, the following hold.*

$$\begin{aligned} \overline{S}_n \cup \underline{S}_2 \cup \cdots \cup \underline{S}_{2\lfloor n/2\rfloor} &= \underline{S}_1 \cup \underline{S}_3 \cup \cdots \cup \underline{S}_{2\lceil n/2\rceil - 1}, \\ \underline{S}_n \cup \overline{S}_2 \cup \cdots \cup \overline{S}_{2\lfloor n/2\rfloor} &= \overline{S}_1 \cup \overline{S}_3 \cup \cdots \cup \overline{S}_{2\lceil n/2\rceil - 1}. \end{aligned}$$

Proof. First we assume that α_i are distinct. Then we may assume that $\alpha_1 < \cdots < \alpha_n$. The term α_i $(1 \le i \le n)$ appears $\binom{n-i}{j-1}$ times in $\underline{S}_j$. Then

$$\delta_{n,i} + \binom{n-i}{1} + \binom{n-i}{3} + \cdots + \binom{n-i}{2\lceil (n-i)/2\rceil - 1}$$

times on the left side of the first equation and

$$\binom{n-i}{0} + \binom{n-i}{2} + \cdots + \binom{n-i}{2\lfloor (n-i)/2\rfloor}$$

times on the right. These numbers are easily seen to be equal. These prove the first equation.

Next we prove the case when the multiset $[\alpha_1, \ldots, \alpha_n]$ contains repetitions. We may assume that

$$\begin{aligned} &\alpha_1 = \ldots = \alpha_{i_1} < \alpha_{i_1+1} = \ldots = \alpha_{i_2} \\ &< \alpha_{i_2+1} = \ldots = \alpha_{i_p} < \alpha_{i_p+1} = \ldots = \alpha_n. \end{aligned}$$

Let

$$T' = \{\beta_1 < \ldots < \beta_{i_1} < \beta_{i_1+1} < \ldots < \beta_{i_2}$$
$$< \beta_{i_2+1} < \ldots < \beta_{i_p} < \beta_{i_p+1} < \ldots < \beta_n\}$$

be another totally ordered set. If we define a map $\varphi : T' \longrightarrow T$ by $\varphi(\beta_i) = \alpha_i$, then φ commutes with $\vee$ and $\wedge$:

$$\varphi(\beta_i \vee \beta_j) = \varphi(\beta_i) \vee \varphi(\beta_j), \quad \varphi(\beta_i \wedge \beta_j) = \varphi(\beta_i) \wedge \varphi(\beta_j).$$

Hence φ commutes with $\overline{S}$ and $\underline{S}$ also. Then the formulae for β_i follow from those for α_i, which are proved in the above.

The second equality follows by taking the dual order. ∎

3. Formulae for Dedekind domains

Let A be a commutative ring (not necessarily with unity). The set of its ideals form a lattice with respect to the order of inclusion. We can define the *greatest common divisor* (GCD) and the *least common multiple* (LCM) for a finite multiset $[\mathfrak{a}_1, \ldots, \mathfrak{a}_n]$ of ideals of A respectively by

$$GCD(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) := \mathfrak{a}_1 \vee \cdots \vee \mathfrak{a}_n = \mathfrak{a}_1 + \cdots + \mathfrak{a}_n,$$
$$LCM(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) := \mathfrak{a}_1 \wedge \cdots \wedge \mathfrak{a}_n = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n.$$

Let us put

$$G(k) := G(k; \mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \prod_{(i_1,\ldots,i_k) \in \binom{N(n)}{k}} (\mathfrak{a}_{i_1} + \cdots + \mathfrak{a}_{i_k}),$$

$$L(k) := L(k; \mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \prod_{(i_1,\ldots,i_k) \in \binom{N(n)}{k}} (\mathfrak{a}_{i_1} \cap \cdots \cap \mathfrak{a}_{i_k}).$$

Let us consider the following equalities.

$$(*)_n \qquad G(n)L(2)L(4) \cdots L(2\lfloor n/2 \rfloor) = L(1)L(3) \cdots L(2\lceil n/2 \rceil - 1),$$

$$(**)_n \qquad L(n)G(2)G(4) \cdots G(2\lfloor n/2 \rfloor) = G(1)G(3) \cdots G(2\lceil n/2 \rceil - 1).$$

We see the following by **2.1**.

Theorem 3.1. *Let A be a commutative ring. If the ideal lattice of A form a totally ordered set, the equalities $(*)_n$ and $(**)_n$ hold.*

Let A be an integral domain and K the field of all fractions of A. (An integral domain is always assumed to be with unity in this paper.) A subset $\mathfrak{a} \subset K$ is called *fractional ideal* if it is an A-submodule of K and if there exists $r \in A \setminus \{0\}$ such that $r\mathfrak{a} \subset A$. If $\mathfrak{a}$ is a fractional ideal, $A\mathfrak{a} = \mathfrak{a}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals of K, we can define their products to be the set $\mathfrak{a}\mathfrak{b}$ of all finite sums of products of elements of $\mathfrak{a}$ and $\mathfrak{b}$.

An integral domain A is called a *Dedekind domain* if the set of all the non-zero fractional ideals form a group with respect to this multiplication. Any ideal of a Dedekind domain can be expressed as an product of prime ideals uniquely up to order (see Lang [2]). For a real number k, let $\lceil k \rceil$ (resp. $\lfloor k \rfloor$) denote the smallest integer that is greater or equal to (resp. the greatest integer that is smaller or equal to) k.

It is known that $(*)_n$ and $(**)_n$ hold for all $n \in \mathbb{N}$ for the ring $\mathbb{Z}$ of rational integers (see the site Wolfman Research [4], [5]). We can say a little more.

Theorem 3.2. *If A is a Noetherian integral domain, the following conditions are equivalent.*

(1) *A is a Dedekind domain*
(2) *The condition $(*)_n$ and $(**)_n$ hold for all multisets $[\mathfrak{a}_1, \dots, \mathfrak{a}_n]$ of ideals of A.*
(3) *The condition $(*)_2 = (**)_2$ holds for all pairs $[\mathfrak{a}_1, \mathfrak{a}_2]$ of ideals of A.*

Proof. (2)$\Longrightarrow$(3): Trivial.

(1)$\Longrightarrow$(2): Suppose that the power of a prime ideal $\mathfrak{p}$ is just α_i in the prime product decomposition of $\mathfrak{a}_i$ $(i = 1, \dots, n)$. We use the notations $\overline{S}_k$ and $\underline{S}_k$ at the beginning of §2, considering $\mathbb{R}$ a totally ordered set with respect to the ordinary order. Then the power of $\mathfrak{p}$ of the left side of the first equality is the sum of the multiset

$$\underline{S}_n(\alpha_1, \dots, \alpha_{2m}) \cup \overline{S}_2(\alpha_1, \dots, \alpha_n) \cup \overline{S}_4(\alpha_1, \dots, \alpha_n) \cup \dots \cup \overline{S}_{2\lfloor n/2 \rfloor}(\alpha_1, \dots, \alpha_n)$$

and one on the right side is the sum of

$$\overline{S}_1(\alpha_1, \dots, \alpha_n) \cup \overline{S}_3(\alpha_1, \dots, \alpha_n) \cup \dots \cup \overline{S}_{2\lceil n/2 \rceil - 1}(\alpha_1, \dots, \alpha_n).$$

These are equal by **2.1**. This proves $(*)_n$. The proof of $(**)_n$ is similar.

(3)$\Longrightarrow$(1): Let $\mathfrak{a} \subsetneqq A$ be a non-zero ideal of A. It is enough to prove that it is the product of a finite number of prime ideals (see Matsumura [3], **11.6**). Let $\mathfrak{p}$ be an associated prime of A-module $A/\mathfrak{a}$. Note that such an associated prime exists because A is Noetherian. Then there exists a non-zero element $\bar{x} := x + \mathfrak{a} \in A/\mathfrak{a}$ such that $\mathfrak{p}$ is the annihilator $\mathrm{Ann}_A(\bar{x})$ of $\bar{x}$. Since $xy \in \mathfrak{a}$ if and only if $y \in \mathrm{Ann}_A(\bar{x}) = \mathfrak{p}$ for $y \in A$, we have $xA \cap \mathfrak{a} = x\mathfrak{p}$. Hence it follows from the assumption (3) that

$$x\mathfrak{p}(xA + \mathfrak{a}) = (xA \cap \mathfrak{a})(xA + \mathfrak{a}) = x\mathfrak{a}.$$

Since A is an integral domain, we have $\mathfrak{p}(xA + \mathfrak{a}) = \mathfrak{a}$. If we put $\mathfrak{a}_1 := xA + \mathfrak{a}$, we have $\mathfrak{a} = \mathfrak{p}\mathfrak{a}_1$. By the condition $\bar{x} \neq 0$, we have $\mathfrak{a} \subsetneqq \mathfrak{a}_1$. If $\mathfrak{a}_1 \subsetneqq A$, applying the arguments above to the ideal $\mathfrak{a}_1$ instead of $\mathfrak{a}$, we obtain a prime ideal $\mathfrak{p}_2$ and an ideal $\mathfrak{a}_2$ with $\mathfrak{a}_1 \subset \mathfrak{a}_2$ such that $\mathfrak{a}_1 = \mathfrak{p}_1\mathfrak{a}_2$, which implies $\mathfrak{a} = \mathfrak{p}_1\mathfrak{a}_1\mathfrak{a}_2$. Continuing this, we obtain sequences of ideals $\mathfrak{a} \subsetneqq \mathfrak{a}_1 \subsetneqq \mathfrak{a}_2 \subsetneqq \cdots$ and prime ideals $\mathfrak{p} = \mathfrak{p}_0,\ \mathfrak{p}_1,\ \mathfrak{p}_2, \dots$ such that $\mathfrak{a} = \mathfrak{p}_0\mathfrak{p}_1 \cdots \mathfrak{p}_i\mathfrak{a}_{i+1}$ for $0 \leq i \leq n-1$. Since A is Noetherian, there exists a number $n \geq 1$ such that $\mathfrak{a}_n = A$ and $\mathfrak{a} = \mathfrak{p}_0\mathfrak{p}_1 \cdots \mathfrak{p}_{n-1}$. ∎

Corollary 3.3. *If A is a Noetherian factorial (UFD) domain, the following conditions are equivalent.*

(1) *A is a principal ideal domain (PID).*
(2) *A is a Dedekind domain*
(3) *The condition $(*)_n$ and $(**)_n$ hold for all the multiset of ideals.*
(4) *The condition $(*)_2 = (**)_2$ holds for all the multiset of two ideals.*

Proof. Implication (1)$\Longleftrightarrow$(2) is obvious from Matsumura [3], **11.6**, **20.1**. The conditions (2), (3) and (4) are equivalent by **3.2**. ∎

4. Examples

Here we show that the equalities stated in the previous section for ideals of a Dedekind domain fail in a general setting using easy examples.

Example 4.1. (1) Let us put $\mathfrak{a}_1 := \langle x^2y\rangle$ and $\mathfrak{a}_2 := \langle xy^2\rangle$ in the polynomial ring $\mathbb{R}[x,y]$. Then

$$L(2)G(2) = (\mathfrak{a}_1 \cap \mathfrak{a}_2)(\mathfrak{a}_1 + \mathfrak{a}_2) = \langle x^4y^3, x^3y^4\rangle$$
$$\subsetneqq \langle x^3y^3\rangle = G(1) = L(1).$$

(2) Let us put $\mathfrak{a}_1 := \langle x\rangle$, $\mathfrak{a}_2 := \langle y\rangle$, $\mathfrak{a}_3 := \langle z\rangle$ in the polynomial ring $\mathbb{R}[x,y,z]$. Then we have

$$G(3)L(2) = \langle x,y,z\rangle\langle xyz\rangle^2 \subsetneqq \langle xyz\rangle^2 = L(1)L(3)$$

(3) Let us put $\mathfrak{a}_1 := \langle x,y\rangle$, $\mathfrak{a}_2 := \langle y,z\rangle$, $\mathfrak{a}_3 := \langle z,x\rangle$ in the polynomial ring $\mathbb{R}[x,y,z]$. Then we have

$$G(3)L(2) = \langle x,y,z\rangle\langle x,yz\rangle\langle y,zx\rangle\langle z,xy\rangle$$
$$\supsetneqq \langle x,y\rangle\langle y,z\rangle\langle z,x\rangle\langle xy,yz,zx\rangle = L(1)L(3),$$

because $x^2yz \in G(3)L(2) \setminus L(1)L(3)$.

(4) Let us put $\mathfrak{a}_1 := \langle x^2yz\rangle$, $\mathfrak{a}_2 := \langle xy^2z\rangle$, $\mathfrak{a}_3 := \langle xyz^2\rangle$ in the polynomial ring $\mathbb{R}[x,y,z]$. Then we have

$$L(3)G(2) = (\mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \mathfrak{a}_3)(\mathfrak{a}_1 + \mathfrak{a}_2)(\mathfrak{a}_2 + \mathfrak{a}_3)(\mathfrak{a}_3 + \mathfrak{a}_1)$$
$$= \langle xyz\rangle^2 \cdot \langle xyz\rangle^3 \cdot \langle x,y\rangle \cdot \langle y,z\rangle \cdot \langle z,x\rangle$$
$$= \langle xyz\rangle^5 \cdot \langle x,y\rangle \cdot \langle y,z\rangle \cdot \langle z,x\rangle,$$
$$G(1)G(3) = (\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3)(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3)$$
$$= \langle xyz\rangle^4 \cdot \langle xyz\rangle \cdot \langle x,y,z\rangle = \langle xyz\rangle^5 \cdot \langle x,y,z\rangle$$

Hence $L(3)G(2) \subsetneqq G(1)G(3)$.

Dedekind domains are Noetherian integral domains. Let us see that integrality and Noetherianity are indispensable in **3.2**. First we show an example of a ring with zero-divisors which satisfy the condition $(*)_n$ and $(**)_n$.

Example 4.2. The commutative ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a Noetherian ring but not an integral domain. There are 4 ideals. It is easy to confirm the formulae $(*)_n$ and $(**)_n$.

Next we show a non-Noetherian local integral domain which satisfy the condition $(*)_2$ and $(**)_2$.

Example 4.3. Let $A := k[x_1, x_2, \ldots]/\langle x_2^2 - x_1, x_3^2 - x_2, \ldots\rangle$ be the quotient ring of a polynomial ring in a countable number of variables over field k. This is an integral domain. Let $\bar{f} \in A$ denote the equivalence class of $f \in k[x_1, x_2, \ldots]$ of f modulo $\langle x_2^2 - x_1, x_3^2 - x_2, \ldots\rangle$. Then A is a local ring with the maximal ideal $\mathfrak{m} := \langle \bar{x}_1, \bar{x}_2, \ldots\rangle$. Take the localization $B := A_{\mathfrak{m}}$. Each non-zero ideal of B is either of the forms

$$I_+(a) := \langle x_n^l :\ n, l \in \mathbb{N},\ l > 2^{n-1}a\rangle \text{ for some } a \in \mathbb{R} \text{ with } a \geq 0$$

or

$$I(l/2^{n-1}) := \langle x_n^l\rangle \text{ for some } n \in \mathbb{N} \text{ and for some } l \in \mathbb{Z} \text{ with } l \geq 0.$$

The ideal $I_+(a)$ is not finitely generated. Hence B is not Noetherian. It is obvious that

$$a > b \Longrightarrow I(a) \subsetneqq I(b),$$

and

$$I_+(l/2^{n-1}) \subsetneqq I(l/2^{n-1}) \quad (n \in \mathbb{N},\ l \in \mathbb{Z},\ l \geq 0).$$

Let $\mathbb{R} \times \{0, \epsilon\}$ be the totally ordered set defined as the direct product, with the lexicographical order, of $\mathbb{R}$ with usual order and $\{0, \epsilon\}$ with $0 < \epsilon$. Then the set of ideals of B with the order of inclusion is isomorphic to a subset of $\mathbb{R} \times \{0, \epsilon\}$ and hence it is totally ordered and the equality $(*)_n$ and $(**)_n$ hold by **3.1**.

5. Inclusion formula for general commutative rings

In this section we assume that A is a commutative ring (not necessarily with unity). We have the following.

Theorem 5.1. *Suppose that $n \leq 6$. Take a multiset $[\mathfrak{a}_1, \ldots, \mathfrak{a}_n]$ of ideals of A. Then we have the following.*

$$(\dagger)_n \qquad L(n)G(2)G(4)\cdots G(2\lfloor n/2 \rfloor) \subset G(1)G(3)\cdots G(2\lceil n/2 \rceil - 1)$$

Proof. In the expressions below, dots, underlines and overlines have no meaning. They are used only for the sake of description.

(1) Proof of $(\dagger)_1$ and $(\dagger)_2$ are very easy.

(2) Proof of $(\dagger)_3$: $L(3)G(2) \subset G(1)G(3)$;

We have only to prove:

$$(\mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{c}) \cdot (\mathfrak{a} + \mathfrak{b})(\mathfrak{a} + \mathfrak{c})(\mathfrak{b} + \mathfrak{c}) \subset \mathfrak{abc}(\mathfrak{a} + \mathfrak{b} + \mathfrak{c})$$

By the symmetry, we have only to prove that $(\mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{c})\mathfrak{a}^2\mathfrak{b}$ and $(\mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{c})\mathfrak{abc}$ are contained on the right side. This is trivial.

(3) Proof of $(\dagger)_4$: $L(4)G(2)G(4) \subset G(1)G(3)$;

We have only to prove:

$$(\underline{\mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{c} \cap \mathfrak{d}})(\mathfrak{a} + \mathfrak{b})(\mathfrak{a} + \mathfrak{c})(\mathfrak{a} + \mathfrak{d})(\dot{\mathfrak{b}} + \mathfrak{c})(\mathfrak{b} + \mathfrak{d})(\underline{\mathfrak{c} + \mathfrak{d}}) \cdot (\dot{\mathfrak{a}} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d})$$

$$\subset \dot{\mathfrak{a}}\dot{\mathfrak{b}}\overline{\mathfrak{c}\mathfrak{d}}(\mathfrak{a} + \mathfrak{b} + \mathfrak{c})(\mathfrak{a} + \mathfrak{b} + \mathfrak{d})(\mathfrak{a} + \mathfrak{c} + \mathfrak{d})(\mathfrak{b} + \mathfrak{c} + \mathfrak{d}).$$

By symmetry, we may replace $\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}$ by $\mathfrak{a}$. Since this replacement is symmetric with respect to $\mathfrak{b}$ and $\mathfrak{c}$, we may replace it by $\mathfrak{b}$. It is obvious that

$$(\mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{c} \cap \mathfrak{d})(\mathfrak{c} + \mathfrak{d}) \subset \mathfrak{c}\mathfrak{d}$$

(underlined parts). Thus the inclusion reduces to the obvious

$$(\mathfrak{a} + \mathfrak{b})(\mathfrak{a} + \mathfrak{c})(\mathfrak{a} + \mathfrak{d})(\mathfrak{b} + \mathfrak{d})$$

$$\subset (\mathfrak{a} + \mathfrak{b} + \mathfrak{c})(\mathfrak{a} + \mathfrak{b} + \mathfrak{d})(\mathfrak{a} + \mathfrak{c} + \mathfrak{d})(\mathfrak{b} + \mathfrak{c} + \mathfrak{d}).$$

(4) Proof of $(\dagger)_5$: $L(5)G(2)G(4) \subset G(1)G(3)G(5)$;

We have only to prove:

$$\begin{aligned}
&(\overline{\mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{c} \cap \mathfrak{d} \cap \mathfrak{e}}) \\
&\cdot(\mathfrak{a} + \mathfrak{b})(\mathfrak{a} + \mathfrak{c})(\mathfrak{a} + \mathfrak{d})(\mathfrak{a} + \mathfrak{e}) \\
&\cdot\underline{(\mathfrak{b} + \mathfrak{c})(\mathfrak{b} + \mathfrak{d})(\mathfrak{b} + \mathfrak{e})}(\dot{\mathfrak{c}} + \mathfrak{d})(\hat{\mathfrak{c}} + \hat{\mathfrak{e}})(\overline{\mathfrak{d} + \mathfrak{e}}) \\
&\cdot(\dot{\mathfrak{a}} + \mathfrak{b} + \mathfrak{c} + \mathfrak{d})(\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{e})(\mathfrak{a} + \mathfrak{b} + \mathfrak{d} + \mathfrak{e})(\mathfrak{a} + \mathfrak{c} + \mathfrak{d} + \mathfrak{e}) \\
&\cdot(\dot{\mathfrak{b}} + \mathfrak{c} + \mathfrak{d} + \mathfrak{e})
\end{aligned}$$

$$
\begin{aligned}
&\subset \dot{\mathfrak{a}}\dot{\mathfrak{b}}\dot{\mathfrak{c}}\overline{\mathfrak{d}\mathfrak{e}} \\
&\cdot(\mathfrak{a}+\mathfrak{b}+\mathfrak{c})(\mathfrak{a}+\mathfrak{b}+\mathfrak{d})(\mathfrak{a}+\mathfrak{b}+\mathfrak{e})(\mathfrak{a}+\mathfrak{c}+\mathfrak{d})(\mathfrak{a}+\mathfrak{c}+\mathfrak{e}) \\
&\cdot(\mathfrak{a}+\mathfrak{d}+\mathfrak{e})\underline{(\mathfrak{b}+\mathfrak{c}+\mathfrak{d})(\mathfrak{b}+\mathfrak{c}+\mathfrak{e})(\mathfrak{b}+\mathfrak{d}+\mathfrak{e})}(\hat{\mathfrak{c}}+\hat{\mathfrak{d}}+\hat{\mathfrak{e}}) \\
&\cdot(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+\mathfrak{e}).
\end{aligned}
$$

We may replace the factor $(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d})$ on the left by $\mathfrak{a}$ without loss of generality. Since this choice is symmetric with respect to $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e})$, we may replace the factor $(\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+\mathfrak{e})$ factor by $\mathfrak{b}$ without loss of generality. Similarly we may replace $(\mathfrak{c}+\mathfrak{d})$ by $\mathfrak{c}$. Thus the dotted $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ on the left are included in the dotted factors on the right. The product of the overlined (resp. underlined, hatted) parts on the left is included in the overlined (resp. underlined, hatted) factor on the right.

Thus we have only to prove:

$$
\begin{aligned}
&(\mathfrak{a}+\mathfrak{b})(\mathfrak{a}+\mathfrak{c})(\mathfrak{a}+\mathfrak{d})(\mathfrak{a}+\mathfrak{e}) \\
&\cdot(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{e})(\mathfrak{a}+\mathfrak{b}+\mathfrak{d}+\mathfrak{e})(\mathfrak{a}+\mathfrak{c}+\mathfrak{d}+\mathfrak{e}) \\
&\quad\subset (\mathfrak{a}+\mathfrak{b}+\mathfrak{c})(\mathfrak{a}+\mathfrak{b}+\mathfrak{d})(\mathfrak{a}+\mathfrak{b}+\mathfrak{e}) \\
&\qquad\cdot(\mathfrak{a}+\mathfrak{c}+\mathfrak{d})(\mathfrak{a}+\mathfrak{c}+\mathfrak{e})(\mathfrak{a}+\mathfrak{d}+\mathfrak{e}) \\
&\qquad\qquad\cdot(\mathfrak{a}+\mathfrak{b}+\mathfrak{c}+\mathfrak{d}+\mathfrak{e}).
\end{aligned}
$$

If we put

$$\mathfrak{b}' := \mathfrak{a}+\mathfrak{b},\ \mathfrak{c}' := \mathfrak{a}+\mathfrak{c},\ \mathfrak{d}' := \mathfrak{a}+\mathfrak{d},\ \mathfrak{e}' := \mathfrak{a}+\mathfrak{e}$$

this reduced to

$$
\begin{aligned}
&\underline{\mathfrak{b}'\mathfrak{c}'\mathfrak{d}'}\mathfrak{e}'(\mathfrak{b}'+\mathfrak{c}'+\mathfrak{e}')(\mathfrak{b}'+\mathfrak{d}'+\mathfrak{e}')(\mathfrak{c}'+\mathfrak{d}'+\mathfrak{e}') \\
&\subset \underline{(\mathfrak{b}'+\mathfrak{c}')(\mathfrak{b}'+\mathfrak{d}')}(\mathfrak{b}'+\mathfrak{e}')\underline{(\mathfrak{c}'+\mathfrak{d}')}(\mathfrak{c}'+\mathfrak{e}')(\mathfrak{d}'+\mathfrak{e}')\cdot(\mathfrak{b}'+\mathfrak{c}'+\mathfrak{d}'+\mathfrak{e}').
\end{aligned}
$$

The inclusion

$$\mathfrak{b}'\mathfrak{c}'\mathfrak{d}' \subset (\mathfrak{b}'+\mathfrak{c}')(\mathfrak{b}'+\mathfrak{d}')(\mathfrak{c}'+\mathfrak{d}')$$

is obvious. If we put

$$\mathfrak{b}'' := \mathfrak{b}+\mathfrak{e},\ \mathfrak{c}'' := \mathfrak{c}+\mathfrak{e},\ \mathfrak{d}'' := \mathfrak{d}+\mathfrak{e},$$

we have only to prove that

$$\mathfrak{e}'(\mathfrak{b}''+\mathfrak{c}'')(\mathfrak{b}''+\mathfrak{d}'')(\mathfrak{c}''+\mathfrak{d}'') \subset \mathfrak{b}''\mathfrak{c}''\mathfrak{d}''(\mathfrak{b}''+\mathfrak{c}''+\mathfrak{d}'').$$

Since $\mathfrak{e}' \subset \mathfrak{b}'' \cap \mathfrak{c}'' \cap \mathfrak{d}''$, this reduces to the case $n = 4$.

(5) Proof of $(\dagger)_6$ is done by computer soft *Mathematica*. The program is shown in the next section. ∎

It is natural to conjecture the following.

Conjecture 5.2. The theorem above holds for all natural numbers n.

6. The proof of the inclusion formula for six ideals.

Here we check validity of **5.1** in the case of six ideals using the computer program on *Mathematica*.

Observing the proof of the case of $n \leq 5$ in the previous section, we notice that we have only to treat the polynomials in $\mathbb{Z}[x_1, \ldots, x_k]$ obtained by considering the ideals as variables.

Hence we put

$$G(k) := G(k; x_1, \ldots, x_6) = \prod_{(i_1,\ldots,i_k)\in\binom{N(6)}{k}} (x_{i_1} + \cdots + x_{i_k}).$$

Let P and Q_0 denote the set of all the monomials appearing in the expansion of $G(2)G(4)G(6)$ and $G(1)G(3)G(5)$ respectively. The problem reduces to validity of the following:

For any $m \in P$ there exists $i \in N(6)$ such that $x_i m \in Q_0$.

If Q denote the set of all the monomials obtained by dividing the elements of Q_0 by some x_i:

$$Q := \{m/x_i : \ m \in Q_0, \ i \in N(6)\},$$

the assertion reduces to the simple inclusion $P \subset Q$. Of course this reduces further to the membership problem of multi-exponents. The program runs as follows. "In[ ]:=" and "Out[ ]:=" mean input and output respectively and $(a, b, c, d, e, f) = (x_1, x_2, x_3, x_4, x_5, x_6)$.

```
In[1] := p = {a, b, c, d, e, f};
In[2] := q1 = Apply[Times, p]
Out[2] := a b c d e f
In[3] := q2 = Product[p[[i]]+p[[j]], {i, 1, 5}, {j, i+1, 6}]
Out[3] := (a+b) (a+c) (b+c) (a+d) (b+d) (c+d) (a+e)
(b+e) (c+e) (d+e) (a+f) (b+f) (c+f) (d+f) (e+f)
In[4] := q3 = Product[p[[i]]+p[[j]]+p[[k]], {i, 1, 4}, {j, i+1, 5}, {k, j+1, 6}]
Out[4] := (a+b+c) (a+b+d) (a+c+d) (b+c+d) (a+b+e) (a+c+e)
(b+c+e) (a+d+e) (b+d+e) (c+d+e) (a+b+f) (a+c+f) (b+c+f)
(a+d+f) (b+d+f) (c+d+f) (a+e+f) (b+e+f) (c+e+f) (d+e+f)
In[5] := q4 = Product[Apply[Plus, Delete[p, {{i},{j}}]], {i, 1, 5}, {j, i+1, 6}]
Out[5] := (a+b+c+d) (a+b+c+e) (a+b+d+e) (a+c+d+e) (b+c+d+e)
(a+b+c+f) (a+b+d+f) (a+c+d+f) (b+c+d+f) (a+b+e+f)
(a+c+e+f) (b+c+e+f) (a+d+e+f) (b+d+e+f) (a+b+e+f)
In[6] := q5 = Product[Apply[Plus, Delete[p, {i}]], {i, 1, 6}]
Out[6] := (a+b+c+d+e) (a+b+c+d+f) (a+b+c+e+f)
(a+b+d+e+f) (a+c+d+e+f) (b+c+d+e+f)
In[7] := q6 = Apply[Plus, p]
Out[7] := a+b+c+d+e+f
In[8] := expo[mon_] := {Exponent[mon, a], Exponent[mon, b],
Exponent[mon, c], Exponent[mon, d], Exponent[mon, e], Exponent[mon, f]}
In[9] := lhs = ExpandAll[q2 q4 q6];
In[10] := rhs = ExpandAll[q1 q3 q5];
In[11] := leftlist = Table[expo[lhs[[k]], {k, 1, Length[lhs]}];
In[12] := rightlist = Table[expo[rhs[[k]], {k, 1, Length[rhs]}];
In[13] := ee[j_] := ee[j] = IdentityMatrix[6][[j]]
In[14] := rightlist0 = {}
In[15] := Do[rightlist0 = Union[rightlist0, {rightlist[[k]]-ee[j]}],
{k, 1, Length[rhs]}, {j, 1, 6}]
In[16] := temp = {};
In[17] := Do[If[MembreQ[rightlist0, leftlist[[k]]]] == False,
AppendTo[temp, leftlist[[k]]], {k, 1, Length[lhs]}]
In[18] := Length[temp]
Out[18] := 0
```

The final output “0” means that $P \subset Q$, proving the case $n = 6$ of the theorem. ∎

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